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On the Best Compact Approximation Problem for Operators between L_p-Spaces

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We construct (for $1<p<2$) an operator from l_p into L_p which has no nearest compact operator. We also give a sufficient condition for an operator from l_p into L_p $(2<p<\infty)$ to have a best compact approximant. © 1987 Academic Press, Inc.

1. INTRODUCTION

A pair of Banach spaces, (X, Y), is said to have the *best compact approximation property* (b.c.a.p.) if, for every bounded linear operator $T: X \to Y$, there exists a compact operator $K: X \to Y$ satisfying $\|T-K\| \leqslant \|T-K'\|$ for all compact K'. We say X has the b.c.a.p. if (X, X) does. It is known that l_p has the b.c.a.p. for $1 \leqslant p < \infty$ (see [ABJS; MW]) while l_∞, L_1, L_∞ and $C[0, 1]$ fail to have it [F]. Recently, with an elegant argument, Benyamini and Lin [BL] showed that, for $1<p<\infty$, $p \neq 2$, L_p fails the b.c.a.p. In this paper we extend their result by showing (l_p, L_p) fails the b.c.a.p. for $1<p<2$ (Theorem 4). We use their key lemma (Lemma 3) but the technical details in our case are much more difficult.

In the last section of this paper (Theorem 16) we give a sufficient condition for an operator $T: L_p \to L_p$ $(2<p<\infty)$ to have a *best compact approximant* (b.c.a.). The condition is that T map uniformly bounded weakly null sequences in L_p into norm null sequences. A corollary of this is the known result that (l_p, L_p) has the b.c.a.p. if $2<p<\infty$.

To end this introduction we state without proof two elementary propositions.

PROPOSITION 1. *Let X and Y be reflexive. Then (X, Y) has the b.c.a.p. iff (Y^*, X^*) has the b.c.a.p.*

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PROPOSITION 2. *Let P be a norm one projection from the Banach space Z onto X. Let T be an operator from X into a Banach space Y. Then T has a best approximant in $K(X, Y)$ if TP has a best approximant in $K(Z, Y)$.*

We use standard Banach space notation and terminology as may be found in the book of Lindenstrauss and Tzafriri [LT] and standard probabilistic material as may be found in the book of Chung [C]. $L_p = L_p([0, 1], m)$, where m is Lebesgue measure.

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2. (l_p, L_p) FAILS THE b.c.a.p. $(1 < p < 2)$

We say a pair of Banach spaces, (X, Y), satisfies the *Benyamini–Lin criterion* (B.L.C.) if there exists $\phi: (0, \infty) \to (0, \infty)$ with $\lim_{\varepsilon \to 0} \phi(\varepsilon) = 0$ such that for all $T \in \mathscr{B}(X, Y)$ with

$$1 = d(T, \mathscr{K}(X, Y)) \leqslant \| T \| < 1 + \varepsilon$$

and any $\delta > 0$, there exists $K \in \mathscr{K}(X, Y)$ with $\| T - K \| < 1 + \delta$ and $\| K \| < \phi(\varepsilon)$. Here $\mathscr{B}(X, Y)$ denotes all bounded linear operators from X into Y and $\mathscr{K}(X, Y)$ is the subspace of compact operators.

LEMMA 3. [BL]. *Let X and Y be either L_p or l_p. Then (X, Y) has the B.L.C. iff (X, Y) has the b.c.a.p.*

THEOREM 4. *Let $1 < p < 2$. Then (l_p, L_p) fails the B.L.C. and hence fails the b.c.a.p.*

From Proposition 1 we obtain

COROLLARY 5. *(L_p, l_p) fails the b.c.a.p. for $2 < p < \infty$.*

We shall need a series of lemmas before we can prove Theorem 4.

LEMMA 6. *Let $1 < p < 2$. Then there exists y, a 2-valued mean zero random variable on $[0, 1]$ with $\| y \|_p = 1$ and $\| y - 1 \|_p^p \geqslant 2 + \delta$ for some $\delta > 0$.*

Proof. Let

$$y = -rI_{[0, 1-s)} + r(1-s)\, s^{-1} I_{[1-s, 1]}$$

where $r > 0$, $0 < s < 1$ and I_A denotes the indicator function of the set A. Clearly $\int_0^1 y = 0$. To have $\| y \|_p = 1$ we also require $r^p = [(1-s) + (1-s)^p s^{1-p}]^{-1}$. Let y^+ and y^- be the positive and negative parts of y, respectively. Assume $\| y^+ \|^p = 1 - n^{-1}$. This is accomplished for large n by taking s small. We shall show that if n is sufficiently large, $\| y - 1 \|^p > 2$.

Simple calculation shows $s = [1 + (n-1)^{1/p-1}]^{-1} \equiv (1+k)^{-1}$, where $k = (n-1)^{1/p-1}$. Since for small s (or equivalently, large n) $y^+ = rs^{-1} - r > 1$ on $[1-s, 1]$ and $\| y^+ \| = 1 - n^{-1}$,

$$\| y^+ - I_{[1-s,1]} \|^p = (1-n^{-1})(rs^{-1} - r - 1)^p (rs^{-1} - r)^{-p}. \qquad (1)$$

Now

$$\begin{aligned} r^p &= [1 - (1+k)^{-1} + [1 - (1+k)^{-1}]^p (1+k)^{p-1}]^{-1} \\ &= [k(1+k)^{-1} + k^p(1+k)^{-p}(1+k)^{p-1}]^{-1} \\ &= (1+k)(k+k^p)^{-1} \\ &= (1+k)k^{-1}(1+k^{p-1})^{-1} \\ &= [1 + (n-1)^{1/p-1}][(n-1)^{1/p-1} n]^{-1}. \end{aligned}$$

Thus we see

$$n^{-1} < r^p < 2n^{-1}. \qquad (2)$$

From (2) we have $rs^{-1} - r \geqslant n^{-1/p}(1+k) - 2n^{-1/p}$. Since $(x-1)x^{-1}$ is increasing for $x > 0$, we get

$$\begin{aligned} &(rs^{-1} - r - 1)(rs^{-1} - r)^{-1} \\ &\quad \geqslant [(1+k)n^{-1/p} - 2n^{-1/p} - 1][n^{-1/p}(1+k) - 2n^{-1/p}]^{-1} \\ &\quad = (kn^{-1/p} - n^{-1/p} - 1)\, n^{1/p}(1+k-2)^{-1} \\ &\quad = (k - 1 - n^{1/p})(k-1)^{-1} = 1 - n^{1/p}(k-1)^{-1}. \end{aligned}$$

Thus by this and (1) we have since $p < 2$,

$$\begin{aligned} \| y^+ - I_{[1-s,1]} \|^p &\geqslant (1-n^{-1})(1 - n^{1/p}(k-1)^{-1})^p \\ &\geqslant (1-n^{-1})(1 - n^{1/p}(k-1)^{-1})^2 \\ &\geqslant (1-n^{-1})(1 - 2n^{1/p}(k-1)^{-1}) \\ &= 1 - n^{-1} - 2n^{1/p}(k-1)^{-1} + 2^{1/p-1}(k-1)^{-1}. \end{aligned} \qquad (3)$$

Also by (2) and $p > 1$,

$$\begin{aligned} \| y^- - I_{[0,1-s)} \|^p &= (1-s)(r+1)^p \\ &\geqslant [1 - (1+k)^{-1}][1 + n^{-1/p}]^p \\ &\geqslant [1 - (1+k)^{-1}][1 + n^{-1/p}] \\ &= 1 + n^{-1/p} - (1+k)^{-1} - n^{-1/p}(1+k)^{-1}. \end{aligned} \qquad (4)$$

Combining (3) and (4),

$$\| y-1 \|^p \geqslant 2 + n^{-1/p} + 2n^{1/p-1}(k-1)^{-1}$$
$$- [n^{-1} + 2n^{1/p}(k-1)^{-1} + (1+k)^{-1} + n^{-1/p}(1+k)^{-1}]. \quad (5)$$

For sufficiently large n, $(1+k)^{-1}$ and $(k-1)^{-1}$ behave like $n^{-1/p-1}$. So, for large n, we can ignore some of the terms in the brackets of (5), namely, $(1+k)^{-1}$ and $(1+k)^{-1}n^{-1/p}$, since they are dominated by $n^{-1/p}$. Now $n^{1/p}(k-1)^{-1}$ behaves like $n^{1/p-1/p-1}$ for large n and $-1/p > 1/p - 1/(p-1) = -1/p(p-1)$. Thus $n^{-1/p}$ also dominates $2n^{1/p}(k-1)^{-1}$. Therefore, for sufficiently large n, $\| y-1 \|^p \geqslant 2+\delta$ for dome $\delta > 0$. Of course, δ depends upon n and decreases to 0 as $n \to \infty$. ∎

Our next lemma is due to Rosenthal [R].

LEMMA 7. *Let $1 < p < 2$ and let $(x_i)_{i=1}^n$ be independent mean zero random variables in L_p. Then*

$$\left\| \sum_{i=1}^n x_i \right\|_p \leqslant 2 \left(\sum_{i=1}^n \| x_i \|_p^p \right)^{1/p}.$$

LEMMA 8. *Let $1 < p < 2$ and let y be as in Lemma 6. Let $(x_i)_{i=1}^\infty$ be independent identically distributed random variables with $x_1 = y$. Then for all j and scalars (a_i),*

$$\left\| x_j + \sum_{i \neq j} a_i x_i \right\|^p \leqslant 1 + 2^{p+1} \sum_{i \neq j} | a_i |^p.$$

Proof. We first state (without proof) an elementary inequality.

SUBLEMMA.

$$\text{For any real } x,\ | 1+x |^p \leqslant 1 + px + 2 | x |^p, \quad (6)$$

and

$$\text{if } x \geqslant 0,\ (1+x)^p \leqslant 1 + px + x^p. \quad (7)$$

To prove Lemma 8, we may assume $j=1$ since the x_i's are exchangeable. Let $x_1 = y = -rI_A + sI_B$, where $A = [0, 1-s)$ and $B = [1-s, 1]$. Then

$$\left\| x_1 + \sum_{i \geqslant 2} a_i x_i \right\|^p = | r |^p \int_A \left| 1 - \sum_{i \geqslant 2} a_i r^{-1} x_i \right|^p$$
$$+ | s |^p \int_B \left| 1 + \sum_{i \geqslant 2} a_i s_i^{-1} x_i \right|^p.$$

By (6), this is

$$\leqslant |r|^p \int_A \left(1 - p \sum_{i\geqslant 2} a_i r^{-1} x_i + 2 \left| \sum_{i\geqslant 2} a_i r^{-1} x_i \right|^p \right)$$
$$+ |s|^p \int_B \left(1 + p \sum_{i\geqslant 2} a_i s^{-1} x_i + 2 \left| \sum_{i\geqslant 2} a_i s^{-1} x_i \right|^p \right).$$

Since the x_i's are independent mean zero,

$$\int_A x_i = \int_B x_i = 0 \qquad \text{for} \quad i \geqslant 2.$$

Thus, this is in turn

$$= |r|^p m(A) + |s|^p m(B) + 2 \int \left| \sum_{i\geqslant 2} a_i x_i \right|^p$$
$$\leqslant 1 + 2^{p+1} \sum_{i\geqslant 2} |a_i|^p,$$

where the last inequality follows by Lemma 7. ∎

Let $1 < p < 2$ be fixed and let $\mathscr{K} = \mathscr{K}(l_p, L_p)$. Let $\delta > 0$ be as in Lemma 6 and let $\delta_n = (2n+2)^{-1}\delta$. To prove Theorem 4 it suffices (by Lemma 3) to construct $\alpha > 0$, $\varepsilon_n \downarrow 0$ and operators $S_n: l_p \to L_p$ satisfying:

$$1 = d(S_n, \mathscr{K}) \leqslant \| S_n \| < 1 + \varepsilon_n \tag{8}$$

and

$$\text{if } K \in \mathscr{K} \text{ with } \|K\| < \alpha, \text{ then } \|S_n - K\| \geqslant 1 + \delta_n. \tag{9}$$

We first construct a sequence of (norm one) operators $T_n: l_p \to L_p$. Then we shall define compact operators $K_n: l_p \to L_p$ and set $S_n = T_n + K_n$. Fix $n \in N$ and let $(\tilde{g}_i)$ be a sequence of random variables supported in $[0, (n-1)/n]$ satisfying:

$$\left\| \sum_{j=1}^{\infty} a_j \tilde{g}_j \right\|_p = \left(\sum_{j=1}^{\infty} a_j^2 \right)^{1/2} [(n-1)/n]^{1/p} \tag{10}$$

and

$$\text{for } x \in L_p[0, (n-1)/n], \ \lim_{j\to\infty} \| \tilde{g}_j - x \| \geqslant [(n-1)/n]^{1/p}. \tag{11}$$

To do this, let $(g_j)_{j=1}^{\infty}$ be a sequence of i.i.d. Gaussian random variables on [0, 1] with $\|g_j\|_p = 1$. Thus $\|\sum a_j g_j\|_p = (\sum a_j^2)^{1/2}$. Define

$$\tilde{g}_j(t) = \begin{cases} g_j(tn/(n-1)), & \text{for } t \in [0, (n-1)/n] \\ 0, & \text{otherwise.} \end{cases}$$

Then $\|\tilde{g}_j\|_p = (n-1)/n$ and (10) holds. Also, since (g_j) is a sequence of symmetric i.i.d. random variables, for $x \in L_p$,

$$\lim_j \|g_j - x\| = \lim_j \|g_j + x\|.$$

Thus, $\lim_j 2\|g_j\| \leqslant \lim_j(\|g_j + x\| + \|g_j - x\|) = 2\lim_j \|g_j - x\|$. Therefore, $\lim_j \|g_j - x\| \geqslant \lim_j \|g_j\| = 1$. Equation (11) follows immediately.

Let (x_i) be the sequence of i.i.d. random variables of Lemma 8 (i.e., $x_1 = y$, where y is as in Lemma 6). Let

$$\tilde{x}_i(t) = \begin{cases} x_i[(t-(n-1)/n)\,n], & \text{for } t \in [(n-1)/n, 1] \\ 0, & \text{otherwise.} \end{cases}$$

Thus $\tilde{x}_i$ is just x_i squished into $[(n-1)/n, 1]$, and so $\|\tilde{x}_i\|^p = n^{-1}$ for all i.

Of course $\tilde{x}_i$ and $\tilde{g}_i$ depend upon n (fixed here) but rather than adopt a cumbersome notation we suppress n in the notation.

Define $T_n: l_p \to L_p$ by $T_n(e_1) = 0$ and $T_n(e_j) = \tilde{g}_j + \tilde{x}_j$ for $j \geqslant 2$. Here (e_j) denotes the unit vector basis of l_p. Thus $\|T_n\| \geqslant 1$. We show below (Lemma 11) that for sufficiently large n, $\|T_n\| = 1$ and so (for large n)

$$1 = \lim_j \|T_n(e_j)\| \leqslant d(T_n, \mathscr{K}) \leqslant \|T_n\| = 1.$$

Define $K_n: l_p \to L_p$ by $K_n(n^{-1/p} e_1) = -I_{[1-n^{-1}, 1]}$ and $K_n(e_j) = 0$ for $j \geqslant 2$. Note $\|K_n\| = 1$ and K_n is compact. Let $S_n = T_n + K_n$.

Our next object is to prove $\|T_n\| = 1$ for large n. First we need some simple lemmas.

LEMMA 9. *Let* $1 < p < 2$ *and* $\sum_{j=1}^{\infty} |a_j|^p = 1$. *Let* $\sum_{j=1}^{\infty} |a_j|^2 \geqslant 1 - \varepsilon$. *Then* $\max_j |a_j|^p \geqslant 1 - g(\varepsilon)$, *where* $g(\varepsilon) \to 0$ *as* $\varepsilon \to 0$.

Proof.

$$\begin{aligned} 1 - \varepsilon \leqslant \sum_{j=1}^{\infty} |a_j|^2 &= \sum_{j=1}^{\infty} |a_j|^p \, |a_j|^{2-p} \\ &\leqslant \max_j |a_j|^{2-p} \sum_{j=1}^{\infty} |a_j|^p \\ &= \max_j |a_j|^{2-p}. \end{aligned}$$

Thus $\max_j |a_j|^p \geqslant (1-\varepsilon)^{p/2-p} \equiv 1 - g(\varepsilon)$, ∎

LEMMA 10. *For $1<p<2$, let*

$$f(x)=x[1-\{(1-x)^{2/p}+x^{2/p}\}^{p/2}]^{-1}, \qquad \textit{if} \quad 0<x<\tfrac{1}{2}.$$

Then f is a bounded function.

Proof. If $0<x\leqslant 1$, then $(1-x)^{2/p}+x^{2/p}<(1-x)+x=1$. Thus f is continuous on $(0,\frac{1}{2}]$. By L'Hopital's rule, $\lim_{x\downarrow 0} f(x)=1$. ∎

Notation. Let $1<p<2$ and let $f(x)$ be as in Lemma 10. Let $M=\sup\{f(x)\colon 0<x\leqslant \frac{1}{2}\}$. Let $0<\varepsilon_0<1$ be such that if $\sum|a_j|^p=1$ and $\sum|a_j|^2\geqslant 1-\varepsilon_0$, then $\max_j|a_j|^p\geqslant 2^{-1}$ (Lemma 9).

LEMMA 11. *Let* $n_0=\max\{1+7M, 2^p[1-(1-\varepsilon_0)^{p/2}]^{-1}\}$. *Then if* $n\geqslant n_0$, $\|T_n\|=1$.

Proof. Let $n\geqslant n_0$ be fixed. Let $\sum_{j=2}^\infty |a_j|^p=1$. We must show $\|\sum_{i\geqslant 2} a_i(\tilde g_i+\tilde x_i)\|^p\leqslant 1$. By the exchangeability of $(\tilde g_i+\tilde x_i)_{i=1}^\infty$, we may suppose $|a_2|=\max_i|a_i|$.

Case 1. $\sum_{i=2}^\infty a_i^2\geqslant 1-\varepsilon_0$.

Thus, by our choice of ε_0, $|a_2|^p\equiv 1-\varepsilon\geqslant 2^{-1}$, so $0\leqslant \epsilon\leqslant 2^{-1}$. If $\varepsilon=0$, then $|a_2|=1$ and $a_j=0$ for $j>2$ and the result is clear. If $\varepsilon>0$, by (10) and Lemma 8

$$\begin{aligned}
&\Big\|\sum_{i\geqslant 2} a_i(\tilde g_i+\tilde x_i)\Big\|^p\\
&\quad=\Big\|\sum_{i\geqslant 2} a_i\tilde g_i\Big\|^p+\Big\|\sum_{i\geqslant 2} a_i\tilde x_i\Big\|^p\\
&\quad\leqslant (1-n^{-1})\Big(\sum_{i\geqslant 2} a_i^2\Big)^{p/2}+n^{-1}\Big(|a_2|^p+2^{p+1}\sum_{i>2}|a_i|^p\Big)\\
&\quad\leqslant (1-n^{-1})\Big[|a_2|^2+\max_{i>2}|a_i|^{2-p}\sum_{i>2}a_i^p\Big]^{p/2}\\
&\qquad+n^{-1}(1-\varepsilon+2^{p+1}\varepsilon)\\
&\quad\leqslant (1-n^{-1})[(1-\varepsilon)^{2/p}+\varepsilon^{(2-p)/p}\varepsilon]^{p/2}\\
&\qquad+n^{-1}[1-\varepsilon+2^{p+1}\varepsilon]\\
&\quad=(1-n^{-1})[(1-\varepsilon)^{2/p}+\varepsilon^{2/p}]^{p/2}+n^{-1}[1-\varepsilon+2^{p+1}\varepsilon].
\end{aligned}$$

This last expression is $\leqslant 1$ provided

$$\begin{aligned} n &\geqslant [1-\varepsilon+2^{p+1}\varepsilon-\{(1-\varepsilon)^{2/p}+\varepsilon^{2/p}\}^{p/2}]\cdot[1-\{(1-\varepsilon)^{2/p}+\varepsilon^{2/p}\}^{p/2}]^{-1} \\ &= 1+(2^{p+1}-1)\,\varepsilon[1-\{(1-\varepsilon)^{2/p}+\varepsilon^{2/p}\}^{p/2}]^{-1}. \end{aligned}$$

By the definition of M, this is true provided $n \geqslant 1+7M$.

Case 2. $\sum_{i\geqslant 2} a_i^2 < 1-\varepsilon_0$.

As in Case 1, by Lemma 7,

$$\begin{aligned} \Big\|\sum_{i\geqslant 2} a_k(\tilde{g}_i+\tilde{x}_i)\Big\|^p &= (1-n^{-1})\Big(\sum_{i\geqslant 2} a_i^2\Big)^{p/2} + n^{-1}\Big\|\sum_{i\geqslant 2} a_i x_i\Big\|^p \\ &< (1-n^{-1})(1-\varepsilon_0)^{p/2} + n^{-1}2^p \sum_{i\geqslant 2} |a_i|^p \\ &< (1-\varepsilon_0)^{p/2} + n^{-1}2^p \leqslant 1, \end{aligned}$$

since $n \geqslant n_0 \geqslant 2^p[1-(1-\varepsilon_0)^{p/2}]^{-1}$. ∎

Our next lemma completes the verification of (8).

Lemma 12. *$\|S_n\| \leqslant 1+\varepsilon_n$, where $\varepsilon_n \to 0$ as $n \to \infty$.*

Proof. Let $\sum_{i=1}^{\infty} |a_i|^p = 1$. Then by (10) and Lemma 7,

$$\begin{aligned} \Big\|S_n\Big(\sum_{i=1}^{\infty} a_i e_i\Big)\Big\|^p &= \Big\|\sum_{i=2}^{\infty} a_i\,\tilde{g}_i\Big\|^p \\ &\quad + \Big\|\sum_{i=2}^{\infty} a_i\tilde{x}_i - a_1 n^{1/p} I_{[(n-1)/n,1]}\Big\|^p \\ &\leqslant (n-1)\,n^{-1}\Big(\sum_{i\geqslant 2} |a_i|^2\Big)^{p/2} \\ &\quad + \Big[n^{-1/p}2\Big(\sum_{i\geqslant 2} |a_i|^p\Big)^{1/p} + |a_1|\Big]^p \\ &\leqslant \sum_{i\geqslant 2} |a_i|^p + [|a_1| + 2n^{-1/p}]^p. \end{aligned}$$

Let $R_n = [|a_1| + 2n^{-1/p}]^p$.

Case 1. $|a_1| \leqslant 2n^{-1/p}$.

Then $R_n \leqslant [4n^{-1/p}]^p = 4^p n^{-1}$.

Case 2. $|a_1| > 2n^{-1/p}$.

Then, by (7),

$$\begin{aligned} R_n &= |a_1|^p \left[1 + 2n^{-1/p}|a_1|^{-1}\right]^p \\ &\leqslant |a_1|^p \left[1 + 2pn^{-1/p}|a_1|^{-1} + 2^p n^{-1}|a_1|^{-p}\right] \\ &= |a_1|^p + 2pn^{-1/p}|a_1|^{p-1} + 2^p n^{-1} \\ &\leqslant |a_1|^p + 2pn^{-1/p} + 2^p n^{-1}. \end{aligned}$$

Thus $\|S_n\| \to 1$ as $n \to \infty$. ∎

It remains only to verify (9). We first need two elementary lemmas.

LEMMA 13. *Let $1 \leqslant p < \infty$ and let $\mathscr{F}$ be a norm bounded subset of L_p. Then for all $\varepsilon > 0$ there exists $\alpha_0 > 0$ such that if $\alpha \leqslant \alpha_0$ and $f \in \mathscr{F}$,*

$$\| |f| - \alpha \|_p^p \geqslant \|f\|_p^p - \varepsilon.$$

Proof. We may assume $\|f\|^p > \varepsilon > 0$. For simplicity we assume $p < 2$ (the only case we need, anyway). Then if $\alpha < \varepsilon^{1/p}$,

$$\begin{aligned} \| |f| - \alpha \|^p \geqslant (\|f\| - \alpha)^p &= \|f\|^p (1 - \alpha \|f\|^{-1})^p \\ &\geqslant \|f\|^p (1 - \alpha \|f\|^{-1})^2 \\ &= \|f\|^p (1 - 2\alpha \|f\|^{-1} + \alpha^2 \|f\|^{-2}) \\ &= \|f\|^p - 2\alpha \|f\|^{p-1} + \alpha^2 \|f\|^{p-2} \\ &= \|f\|^p + h(\alpha, f) \end{aligned}$$

where $h(\alpha, f) \to 0$ uniformly for $f \in \mathscr{F}$ with $\|f\|^p \geqslant \varepsilon$ as $\alpha \to 0$. ∎

LEMMA 14. *Let $K: l_p \to L_p$ be a bounded linear operator with $\|K\| \leqslant \eta^{1+1/p}$ for some $\eta > 0$. Then if A is any measurable subset of $[0, 1]$, $|K(e_1)| \leqslant \eta[m(A)]^{-1/p}$ on a subset of A of measure at least $(1 - \eta)\, m(A)$.*

Proof. Let $A_0 = A \cap \{t : |K(e_1)(t)| > \eta[m(A)]^{-1/p}\}$. Then $\eta^{p+1} \geqslant \int_{A_0} |K(e_1)|^p > \eta^p m(A)^{-1} m(A_0)$. Thus $m(A_0) \leqslant \eta m(A)$ or $m(A \setminus A_0) \geqslant (1 - \eta)\, m(A)$. ∎

Our next lemma proves (9) and thus completes the proof of Theorem 4.

LEMMA 15. *Let $\delta_n = (2n+2)^{-1}\delta$, where δ is as in Lemma 6. Then there exists $\eta > 0$ so that for $n \in N$, if $K \in \mathscr{K}(l_p, L_p)$ with $\|K\| \leqslant \eta^{1+1/p}$, then $\|S_n - K\| \geqslant (1 + \delta_n)^{1/p}$.*

Proof.

Claim. There exist $\eta > 0$ such that if K is compact with $\|K\| \leqslant \eta^{1+1/p}$ and $j \geqslant 2$, then

$$\|[\tilde{x}_j - 1 - K(n^{-1/p}e_1)]\, I_{[1-n^{-1},1]}\|^p \geqslant n^{-1}(2+\delta/2). \tag{12}$$

Suppose the claim has been proved. Let $K \in \mathscr{K}(l_p, L_p)$ with $\|K\| \leqslant \eta^{1+1/p}$. Let $z_j = n^{-1/p}e_1 + e_j$. Then $\|Ke_j\| \to 0$ as $j \to \infty$ and so if $B = [1-n^{-1}, 1]$,

$$\|(S_n - K)\, z_j\|^p = \|T_n(e_j) - I_B - K(n^{-1/p}e_1)\|^p + \alpha_j$$

(where $\alpha_j \to 0$ as $j \to \infty$)

$$= \int_B |\tilde{x}_j - I_B - K(n^{-1/p}e_1)|^p + \int_{[0,1]\setminus B} |\tilde{g}_j - K(n^{-1/p}e_1)|^p + \alpha_j.$$

Now by (11) and (12) this is (in the limit as $j \to \infty$)

$$\geqslant n^{-1}(2+\delta/2) + n^{-1}(n-1) = 1 + n^{-1} + n^{-1}(\delta/2).$$

Thus

$$\|S_n - K\|^p \geqslant [1 + n^{-1} + n^{-1}\delta/2][1+n^{-1}]^{-1} = 1 + \delta_n.$$

Proof of Claim. By Lemma 14, if $\|K\| < \eta^{1+1/p}$ then $|K(n^{-1/p}e_1)| \leqslant \eta$ on some subset of $[1-n^{-1}, 1]$ of measure at least $(1-\eta)\, n^{-1}$. Thus to prove (12), it suffices to show that if η is taken sufficiently small and $y \in L_p$ is such that $|y| \leqslant \eta$ on a subset of $[0, 1]$ of measure at least $1 - \eta$ and (x_j) are the random variables of lemma 8, then

$$\|x_j - 1 - y\|^p \geqslant 2 + \delta/2. \tag{13}$$

By Lemma 13 applied to $\mathscr{F} = \{|x_j - 1|\}_{j=1}^{\infty}$ and $\varepsilon = \delta/8$, there exists $\eta_0 > 0$ so that if $0 \leqslant \eta \leqslant \eta_0$ then

$$\begin{aligned}\||x_j - 1| - \eta\|^p &\geqslant \|x_j - 1\|^p - \delta/8 \\ &\geqslant 2 + 7\delta/8.\end{aligned}$$

Furthermore, the set of functions $\{||x_j - 1| - \eta|^p : j \in N,\ 0 \leqslant \eta \leqslant \eta_0\}$ is uniformly integrable (in fact, uniformly bounded) and so there exists $\eta_1 \leqslant \eta_0$ so that if $D \subseteq [0, 1]$ with $m(D) \geqslant 1 - \eta_1$, and $0 \leqslant \eta \leqslant \eta_1$, then

$$\begin{aligned}\|(|x_j - 1| - \eta)|_D\|^p &\geqslant \||x_j - 1| - \eta\|^p - \delta/8 \\ &\geqslant 2 + 3\delta/4.\end{aligned}$$

Let $\eta = \min\{\eta_1, 2^{-1}(\delta/4)^{1/p}\}$. We verify (13).

Let $y \in L_p$; $D = \{t : |y(t)| < \eta\}$ and suppose $m(D) \geqslant 1 - \eta$. Then

$$\|x_j - 1 - y\|^p \geqslant \int_{[|x_j - 1| \geqslant \eta]} |x_j - 1 - y|^p$$

$$\geqslant \int_{[|x_j - 1| \geqslant \eta]} ||x_j - 1| - \eta|^p I_D$$

$$= \int ||x_j - 1| - \eta|^p I_D - \int_{[|x_j - 1| < \eta]} ||x_j - 1| - \eta|^p I_D$$

$$\geqslant 2 + 3\delta/4 - 2^p \eta^p \geqslant 2 + \delta/2. \quad \blacksquare$$

3. A Positive Result

L. Weis [W] has shown that if $1 < p < \infty$ and $T: L_p \to L_p$ is an integral operator satisfying

$$\text{if } (x_n) \text{ is a uniformly bounded weakly null sequence in } L_p, \text{ then } \|Tx_n\| \to 0 \tag{$*$}$$

then T has a best compact approximant. In this section we shall show that if $p > 2$ the assumption that T is integral may be removed.

THEOREM 16. *Let $2 < p < \infty$ and let $T: L_p \to L_p$ be a bounded linear operator. Then if T satisfies $(*)$, T has a best compact approximant.*

Remark 17. By Theorem 4 and Proposition 2, the analogue of Theorem 16 is false for $1 < p < 2$.

The proof of Theorem 16 uses a criterion implicit in the work of Weis [W]. We say a set, $\mathscr{C}$, of bounded operators from X into Y is *closed under compact perturbations* if $T - K \in \mathscr{C}$ for all $T \in \mathscr{C}$ and $K \in \mathscr{K}(X, Y)$.

LEMMA 18. *Let $\mathscr{C}$ be a set of bounded operators from X into Y which is closed under compact perturbations and scalar multiplication. Suppose there exists $0 < \gamma < 1$ and $c < \infty$ so that if $\varepsilon > 0$ and $T \in \mathscr{C}$ is such that $\|T\| = 1 + \varepsilon$ and $d(T, \mathscr{K}(X, Y)) = 1$, then there exists $K \in \mathscr{K}(X, Y)$ with $\|K\| \leqslant c\varepsilon$ and $\|T - K\| \leqslant 1 + \gamma\varepsilon$. Then every $T \in \mathscr{C}$ has a best compact approximant.*

Proof. Let $T \in \mathscr{C}$. We may assume $\|T\| = 1 + \varepsilon$ with $\varepsilon > 0$ and $d(T, \mathscr{K}(X, Y)) = 1$. Choose $K_1 \in \mathscr{K}$ so that $\|K_1\| \leqslant c\varepsilon_1$, where $\varepsilon_1 = \varepsilon$, and $\|T - K_1\| = 1 + \varepsilon_2$ with $\varepsilon_2 \leqslant \gamma\varepsilon_1$. Let $T_1 = T - K_1$. Then $d(T_1, \mathscr{K}) = 1$ and so we may choose $K_2 \in \mathscr{K}$ with $\|K_2\| \leqslant c\varepsilon_2 \leqslant c\gamma\varepsilon_1$ and $\|T_1 - K_2\| \leqslant 1 + \gamma\varepsilon_2 \leqslant 1 + \gamma^2\varepsilon_1$. Continue in this manner. It follows that $\sum_{i=1}^{\infty} K_i$ is absolutely convergent ($\|K_i\| \leqslant c\gamma^{i-1}\varepsilon$) to a compact operator K with $\|T-K\|=1$. ∎

We need two more elementary lemmas.

LEMMA 19. *Let (h_i) be the Haar basis for L_p $(1<p<\infty)$. Let P_n be the basis projection from L_p onto* $\operatorname{span}(h_i)_{i=1}^n$. *Then if I is the identity operator on L_p, there exists $c_p<2$ such that for all n, $\|I-P_n\| \leqslant c_p$.*

Proof. Since $\|P_n\|=1$, $\|I-P_n\| \leqslant 2$. Also $\|I-P_n\|=1$ in L_2 and thus the result follows by interpolation. ∎

LEMMA 20. *Let (y_n) be a weakly null sequence in L_p $(1<p<\infty)$. Suppose $(|y_n|^p)_{n=1}^\infty$ is uniformly integrable and let (k_n) be a subsequence of N. Then both $(|P_{k_n}y_n|^p)$ and $(|(I-P_{k_n})y_n|^p)$ are uniformly integrable and weakly null.*

Proof. It suffices to show $(|P_{k_n}y_n|^p)$ is uniformly integrable. But this follows since each P_{k_n} is a conditional expectation projection (with respect to a finite σ-algebra of dyadic sets in $[0,1]$). Indeed, one can show that for $\delta>0$, $\sup\{\int_A |P_{k_n}y_n|^p : m(A) \leqslant \delta\} \leqslant \sup\{\int_A |y_n|^p : m(A) \leqslant \delta\}$. ∎

Proof of Theorem 16. The class of operators on L_p which satisfy $(*)$ is closed under compact perturbation and scalar multiplication. Thus, by Lemma 18, it suffices to show that if T satisfies $(*)$, $\|T\|=1+\varepsilon$ and $d(T,\mathscr{K})=1$, then there exists a compact operator K with $\|T-K\| \leqslant 1+\gamma\varepsilon$ and $\|K\| \leqslant \varepsilon$.

Let $\eta=\varepsilon(1+\varepsilon)^{-1}$ so that $1-\eta=(1+\varepsilon)^{-1}$. We shall show that $K_n=\eta TP_n$ works if n is sufficiently large and γ is any number larger than γ_p, where $c_p=1+\gamma_p$ is as in Lemma 19. Note $\|K_n\| \leqslant \varepsilon$.

Let $T-K_n=S_n=(1-\eta)\,TP_n+T(I-P_n)$. We must show $\|S_n\| \leqslant 1+\gamma\varepsilon$ for n sufficiently large. To make the following argument clearer we have ignored arbitrarily small errors. Choose $w_n \in L_p$ with $\|w_n\|=1$ and $\|S_n(w_n)\|=\|S_n\|$ (one small error ignored). By passing to subsequences several times (and ignoring the small errors) we may assume we have (k_n), a subsequence of N, so that

$$w_{k_n}=x+x_n,\ P_{k_1}x=x, \text{ and } (x_n) \text{ is weakly null.} \tag{14}$$

$x_n=y_n+z_n$, where $(|y_n|^p)$ is uniformly integrable, (z_n) is a disjointly supported sequence relative to $[0,1]$ and z_n is disjointly supported from $x+y_n$. (15)

$$(y_n) \text{ and } (z_n) \text{ are block bases of } (h_n) \text{ with } P_{k_1}y_n=0 \text{ for all } n. \tag{16}$$

$$\|S_{k_n}(w_{k_n})\|=\|S_{k_n}\|. \tag{17}$$

$$S_{k_n}(y_n)=0 \text{ for all } n. \tag{18}$$

$$Tx \text{ is disjointly supported from both } TP_{k_n}z_n \text{ and } T(I-P_{k_n})\,z_n. \tag{19}$$

All of these may be accomplished by standard subsequence arguments. Result (14) is obtained by letting x be the weak limit of a subsequence of (w_n) and ignoring $(I-P_{k_1})x$. Result (15) follows from the "subsequence splitting lemma" in L_1 applied to $(|x_n|^p)$. Result (18), or actually $\|S_{k_n}(y_n)\| \to 0$ follows from Lemma 20, the definition of S_{k_n} and the fact that $(*)$ implies if (f_n) is weakly null with $(|f_n|^p)$ uniformly integrable, then $\|Tf_n\| \to 0$. (19) follows from the fact that (z_n) may be assumed to be equivalent to the unit vector basis of l_p (if its not norm null) and since $p>2$ its image must have small support (see [KP]). Thus

$$\begin{aligned}
\|S_{k_n}\|^p &\overset{(14),(17)}{=} \|S_{k_n}(x+x_n)\|^p \overset{(15),(18)}{=} \|S_{k_n}(x+z_n)\|^p \\
&\overset{(14)}{=} \|(1-\eta)Tx + (1-\eta)TP_{k_n}z_n + T(I-P_{k_n})z_n\|^p \\
&\overset{(19)}{=} \|(1-\eta)Tx\|^p + \|(1-\eta)TP_{k_n}z_n + T(I-P_{k_n})z_n\|^p \\
&\leqslant \|x\|^p + \|(1-\eta)Tz_n + \eta T(I-P_{k_n})z_n\|^p \\
&\leqslant \|x\|^p + [(1-\eta)\|Tz_n\| + \eta\|T(I-P_{k_n}z_n\|]^p.
\end{aligned}$$

Since $d(T,\mathscr{K})=1$, for large n we have (essentially) $\|Tz_n\| \leqslant \|z_n\|$ and $\|T(I-P_{k_n})z_n\| \leqslant \||I-P_{k_n})z_n\|$. Thus, continuing, using Lemma 19,

$$\begin{aligned}
\|S_{k_n}\|^p &\leqslant \|x\|^p + [(1-\eta)\|z_n\| + \eta(1+\gamma_p)\|z_n\|]^p \\
&\leqslant \|x\|^p + \|z_n\|^p(1+\eta\gamma_p)^p \\
&\overset{(14),(16)}{\leqslant} (1+\eta\gamma_p)^p(\|x+y_n\|^p + \|z_n\|^p) \\
&\overset{(15)}{\leqslant} (1+\eta\gamma_p)^p\|x+y_n+z_n\|^p \\
&= (1+\eta\gamma_p)^p \leqslant (1+\varepsilon\gamma_p)^p. \quad \blacksquare
\end{aligned}$$

Remark 21. It follows from Theorem 16 and Proposition 2 that (l_p, L_p) has the b.c.a.p. for $2<p<\infty$. By Proposition 1, (L_p, l_p) has the b.c.a.p. for $1<p<\infty$. By Theorem 4 and Proposition 1, (L_p, l_p) fails the b.c.a.p. for $2<p<\infty$. It is not known whether (l_2, L_p) or (L_p, l_2) has the b.c.a.p. for $1<p<\infty$, $p\neq 2$.

Also, it is not difficult to show the following spaces have the b.c.a.p.: (l_p, L_q) for $2<p, q<\infty$ or $1<q<2<p<\infty$.

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